Faculty of Science, Technology, Engineering and Mathematics
M303 Further Pure Mathematics

M303 Solutions to the Specimen Examination Paper

Section 1

Question 1

(a) It suffices to consider the 5 residue classes to which n might belong in turn.

$$n \equiv 0 \pmod 5 \Longrightarrow n^5 - n \equiv 0^5 - 0 = 0 \pmod 5$$
$$n \equiv 1 \pmod 5 \Longrightarrow n^5 - n \equiv 1^5 - 1 = 0 \pmod 5$$
$$n \equiv 2 \pmod 5 \Longrightarrow n^5 - n \equiv 2^5 - 2 = 0 \pmod 5$$
$$n \equiv 3 \pmod 5 \Longrightarrow n^5 - n \equiv 3^5 - 3 \equiv (-1)^2.3 - 3 = 0 \pmod 5$$
$$n \equiv 4 \pmod 5 \Longrightarrow n^5 - n \equiv 4^5 - 4 \equiv (-1)^5 + 1 = 0 \pmod 5$$ 2

Alternatively, by Fermat's Little Theorem (alternative version), HB p21, since 5 is a prime, for all $n \in \mathbb{Z}$, $n^5 \equiv n \pmod 5$ and so $n^5 - n \equiv 0 \pmod 5$ for all $n \in \mathbb{Z}$.

(b) We use the Principle of Mathematical Induction as described on HB p14.

Let $P(n)$ be the statement that $a^{2^n} - 1$ is divisible by 2^{n+2}.

Since a is odd, we can write $a = 2k + 1$ for some $k \geq 0$.

For the base case $n = 1$ we have

$$a^{2^1} - 1 = (2k+1)^2 - 1 = 4(k^2 + k).$$

Now, $k^2 + k = k(k+1)$ and so $k^2 + k$ is the product of two consecutive integers and is therefore even. Thus $8 = 2^{1+2}$ divides $a^{2^1} - 1$.

Hence $P(1)$ holds. 3

For the inductive step, suppose $P(n)$ holds for some $n \geq 1$, that is $a^{2^n} - 1$ is divisible by 2^{n+2}. We aim to show that this implies $P(n+1)$ is true.

By completing the square, we see that

$$a^{2^{n+1}} - 1 = (a^{2^n} - 1)(a^{2^n} + 1).$$

Now, since $P(n)$ holds, $a^{2^n} - 1$ is divisible by 2^{n+2} and, in addition, $a^{2^n} + 1$ is even (and in particular divisible by 2). Putting this together, we conclude that $a^{2^{n+1}} - 1$ is divisible by $2 \cdot 2^{n+2} = 2^{(n+1)+2}$, as required. 2

Hence, by the Principle of Mathematical Induction, a^{2^n} is divisible by 2^{n+2} for all $n \geq 1$. 1

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(c) Try a few small values. For $a = 3$ and $a = 5$ then we have

$$a^{2^1} - 1 = (8k \pm 3)^2 - 1 = 16(4k^2 \pm 3k) + 8$$

which is *not* divisible by $16 = 2^{1+3}$. Thus, not even the base case holds here. 1

For $a = 1, 7$ or 9 note that $a^2 - 1$ is divisible by 8. Conjecture that the condition is true if $a = 8k \pm 1$.

Adapting the proof to (b) gives: If $a = 8k \pm 1$, then the base case is

$$a^{2^1} - 1 = (8k \pm 1)^2 - 1 = 16(4k^2 \pm k)$$

which is divisible by $16 = 2^{1+3}$. The rest of the proof by induction follows as in (b). 2

11 Total

Question 2

(a) (i) 495

(ii) 165

(iii) Possible answers are (11,0), (22,0), (44,0), (55,0), (77,0), (88,0). (Note that both (33,0) and (66,0) have order 3 in G.) 3

(b) (i) $\mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_7$

(ii) $n = 210$

(iii) No, G does not contain an element of order 840 — repeated use of Proposition 2.8, HB p32 shows that the greatest order of an element is 210.

(iv) $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7$ (or $\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7$) 5

(c) For each $g_i \in G$ there is a $j \in \{1, 2, \ldots, n\}$ such that $g_j = -g_i$ (since inverses always exist in a group). Therefore writing $-g_i$ instead of g_j we have
$2(g_1 + \cdots + g_n) = g_1 + \cdots + g_n + (-g_1 - \cdots - g_n) = g_1 - g_1 + \cdots g_n - g_n$
(since G is abelian) and the sum is zero. 3

11 Total

Question 3

(a) The divisors are $1, 2, 5, 10, 11, 22, 55, 110$.

(i) $\tau(110) = 8$.

(ii) $\sigma(110) = 1 + 2 + 5 + 10 + 11 + 22 + 55 + 110 = 216$.

(iii) $\phi(110) = 1 \cdot 4 \cdot 10 = 40$. 3

(b) (i) Since $307 \equiv 3 \pmod 8$, $(2/307) = -1$ by Chapter 10 Prop 3.2.

(ii) We have

$$\begin{aligned}
(23/307) &= -(307/23) && \text{by LQR, since } 23 \equiv 307 \equiv 3 \pmod 4 \\
&= -(8/23) && \text{since } 307 \equiv 8 \pmod{23} \\
&= -(4/23)(2/23) && \text{by Thm 2.2(c) HB p46} \\
&= -(2/23) && \text{by Thm 2.2(b) HB p46} \\
&= (-1)(1) = -1. && \text{Thm 2.2(e) and since } 23 \equiv 7 \pmod 8
\end{aligned}$$

3

(c) Reducing modulo 3, we find

$$f(x) \equiv x^4 + 2x^3 + x^2 + x + 2.$$

We first check for roots in $\mathbb{Z}_3$ and find that $f(0) \equiv 2$, $f(1) \equiv 1$, $f(2) \equiv f(-1) \equiv 1 \pmod 3$. So f has no roots in $\mathbb{Z}_3$. We now check for quadratic factors, which we may assume are monic. So suppose that for some $a, b, c, d \in \mathbb{Z}_3$ we have

$$x^4 + 2x^3 + x^2 + x + 2 \equiv (x^2 + ax + b)(x^2 + cx + d).$$

Comparing coefficients, we find

$$\begin{aligned}
a + c &\equiv 2 \pmod 3 && (1) \\
ac + b + d &\equiv 1 \pmod 3 && (2) \\
ad + bc &\equiv 1 \pmod 3 && (3) \\
bd &\equiv 2 \pmod 3. && (4)
\end{aligned}$$

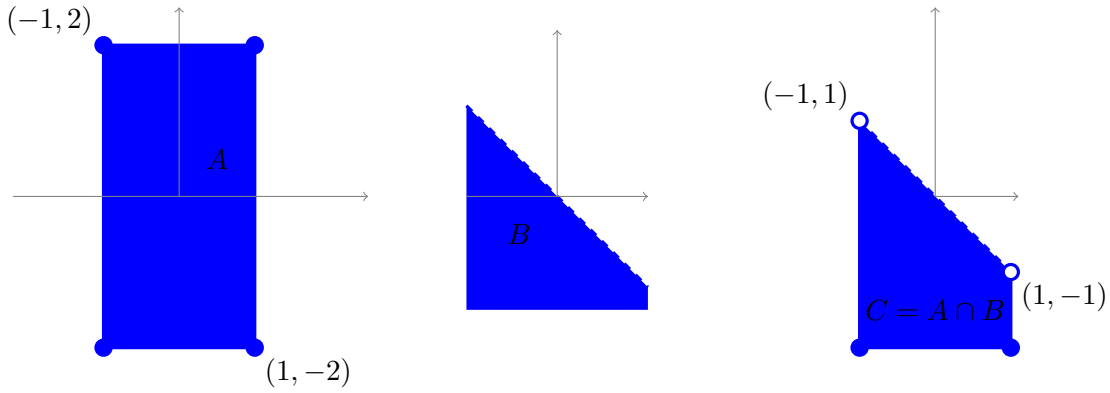
From (4), we deduce either $b \equiv 1$ and $d \equiv 2$ or $b \equiv 2$ and $d \equiv 1$. Either way, $b + d \equiv 0$. Then (1) and (2) become $a + c \equiv 2$ and $ac \equiv 1$, the only solutions to which are $a \equiv c \equiv 1$. But then (3) gives $b + d \equiv 1$, which contradicts our earlier conclusion that $b + d \equiv 0$. (All congruences are modulo 3.) It follow that no such a, b, c, d exist in $\mathbb{Z}_3$.

Thus f is irreducible modulo 3. Hence by Lemma 4.19 HB p 52, f is irreducible in $\mathbb{Z}[x]$ and hence, by Gauss's Lemma (Theorem 4.13 HB p52), irreducible in $\mathbb{Q}[x]$. 5

11 Total

Question 4

(a)



3

Description in words:

A is the filled rectangle with vertices at $(-1,-2)$, $(1,-2)$, $(1,2)$, and $(-1,2)$. The boundary is included.

B consists of all points below the boundary line $x+y=0$ but does not include the line itself.

C is the intersection of these two sets and is the quadrilateral that has vertices $(-1,-2)$, $(1,-2)$, $(1,-1)$. and $(-1,1)$. The only part of the boundary segments that is not included is the segment joining $(1,-1)$. and $(-1,1)$: these vertices are excluded as well.

(b) $(1,-1)$ is a $d^{(2)}$-closure point of C because, for example, the sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n = (1-1/n,-1)$, is a sequence of points from C that $d^{(2)}$-converges to $(1,-1)$, HB p84, Definition 3.6 and Corollary 3.11. 2

(c) The question states 'write down', so we do not need to justify the answers.

$$\begin{aligned}
\mathrm{Cl}_{(\mathbb{R}^2,d^{(2)})}(C) &= \{(a,b)\in\mathbb{R}^2 : |a|\le 1 \text{ and } |b|\le 2 \text{ and } a+b\le 0\}\\
\mathrm{Int}_{(\mathbb{R}^2,d^{(2)})}(C) &= \{(a,b)\in\mathbb{R}^2 : |a|< 1 \text{ and } |b|< 2 \text{ and } a+b< 0\}\\
\mathrm{Bd}_{(\mathbb{R}^2,d^{(2)})}(C) &= \{(-1,b)\in\mathbb{R}^2 : -2\le b\le 1\}\cup\{(a,-2)\in\mathbb{R}^2 : -1\le a\le 1\}\\
&\quad\cup\{(1,b)\in\mathbb{R}^2 : -2\le b\le -1\}\\
&\quad\cup\{(a,b)\in\mathbb{R}^2 : a+b=0 \text{ and } -1\le a\le 1\}.
\end{aligned}$$

4

(d) The point $(-1,1)$ is not a boundary point of C: for any $(a,b)\in C$, we have $b\neq 1$ and so $d((a,b),(-1,1))\ge d_0(b,1)=1$, implying that $B_d((-1,1),1)\cap C=\emptyset$ and the claim immediately follows from HB p86, Definition 5.3. 2

11 Total

Question 5

(a) We have $\alpha^3 = (\sqrt{2})^3 - 3(\sqrt{2})^2 i + 3\sqrt{2}i^2 - i^3$ and so
$\alpha^3 = 2\sqrt{2} - 3 \cdot 2 \cdot i + 3 \cdot \sqrt{2} \cdot (-1) + i = -\sqrt{2} - 5i$. 2

(b) Eliminating $\sqrt{2}$ and i from α and α^3 gives

$\alpha^3 + \alpha = -6i$ and $\alpha^3 - 5\alpha = -6\sqrt{2}$.

But both $\alpha^3 + \alpha$ and $\alpha^3 - 5\alpha$ are in $\mathbb{Q}(\alpha)$ and so $\sqrt{2}, i \in \mathbb{Q}(\alpha)$.
Alternatively, from just one of the above equations conclude that one of $\sqrt{2}, i$ is in $\mathbb{Q}(\alpha)$, and the other one is then in $\mathbb{Q}(\alpha)$ because α is. 2

(c) By (b) one has $\mathbb{Q}(\sqrt{2}, i) \subseteq \mathbb{Q}(\alpha)$, and since $\alpha \in \mathbb{Q}(\sqrt{2}, i)$ it follows that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, i)$. 1

(d) Since $i \notin \mathbb{Q}(\sqrt{2})$ and i is a root of a quadratic polynomial over $\mathbb{Q}(\sqrt{2})$ (in fact, over $\mathbb{Q}$ – also acceptable), we have $[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}(\sqrt{2})] = 2$.

By the KLM Theorem (Theorem 2.29 of Chapter 18) and by (c) we have $[\mathbb{Q}(\alpha) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}(\sqrt{2})] \cdot [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2 \cdot 2 = 4$. 2

(The roles of $\sqrt{2}$ and i are, of course, interchangeable in the above argument.)

(e) As $\alpha \in \mathbb{Q}(\beta)$ and $\beta \in \mathbb{Q}(\alpha)$, one has $[\mathbb{Q}(\beta) : \mathbb{Q}] = 4$. Since $\alpha = 1 - 2\beta$, we find that by squaring one obtains
$(2\beta - 1)^2 = \alpha^2 = (\sqrt{2} - i)^2 = 1 - 2\sqrt{2}i$, that is, $(2\beta - 1)^2 - 1 = -2\sqrt{2}i$.

Squaring again gives $(2\beta - 1)^4 - 2(2\beta - 1)^2 + 1 = -8$, which simplifies to $16\beta^4 - 32\beta^3 + 16\beta^2 + 8 = 0$.

Since by (d) the minimal polynomial for $\beta = \frac{1}{2}(1 - \alpha)$ over $\mathbb{Q}$ must have degree 4, and is by definition monic, the sought minimal polynomial is $x^4 - 2x^3 + x^2 + \frac{1}{2}$. 4

11 Total

Question 6

(a) (i)

$$\begin{aligned} d((0,1/n),(a,b)) &= \max\{2|a|, 3|1/n - b|\} \\ &\geq 3|1/n - b| = 3|b - 1/n|, \end{aligned}$$ 2

as required.

(ii) By part (i), we have seen that $d((0,1/n),(a,b)) \geq 3|b - 1/n|$ for any $(a,b) \in X$. But if $(a,b) \in X$, then $|b| > 0$, and so $(d((0,1/n),(a,b)))$ is not a real null sequence and so $((0,1/n))$ does not d-converge to (a,b). Since (a,b) is an arbitrary point in X, we conclude that $((0,1/n))$ is not d-convergent in X. 2

(iii) Each term of the sequence $((0,1/n))$ lies in X.

If we show that it is a Cauchy sequence, then the result of part (ii) immediately implies that the space is not complete. HB p135 Definition 1.3

Let $\mathbf{a}_n = (0,1/n)$ for $n \in \mathbb{N}$. Then for $n, m \in \mathbb{N}$,

$$\begin{aligned} d(\mathbf{a}_n, \mathbf{a}_m)) &= 3|1/n - 1/m| \\ &\leq 3(1/n + 1/m). \end{aligned}$$

Hence for $\varepsilon > 0$, choose N so that $6/N < \varepsilon$ and then for $m, n > N$, $d(\mathbf{a}_n, \mathbf{a}_m)) < \varepsilon$ 3

and so the sequence is Cauchy.

(iv) If X was sequentially compact, then Theorem 2.3 of Chapter 23 (HB page 136) and Theorem 5.12 of Chapter 22 (HB page 134) would imply that X was complete, which is false. Hence X is not d-sequentially compact. 1

(b) A can be written as the union of three line segments, $A = A_1 \cup A_2 \cup A_3$, where

$$\begin{aligned} A_1 &= \{(-a, 1-a) \in \mathbb{R}^2 : a \in [0,1]\} \\ A_2 &= \{(a,0) \in \mathbb{R}^2 : |a| \leq 1\} \\ A_3 &= \{(a, 1-a) \in \mathbb{R}^2 : a \in [0,1]\}. \end{aligned}$$

Now $A_1 \cap A_2 = \{(-1,0)\}$ and so by HB p122 Lemma 3.15, and the fact that line segments are $d^{(2)}$-connected, we deduce that $A_1 \cup A_2$ is $d^{(2)}$-connected.

But $(A_1 \cup A_2) \cap A_3 = \{(1,0),(0,1)\}$ and so using HB p122 Lemma 3.15 again, we conclude that $(A_1 \cup A_2) \cup A_3 = A_1 \cup A_2 \cup A_3 = A$ is $d^{(2)}$-connected as required. 3

11 Total

Part 2

Question 7

(a) (i) 4

(ii) Let m_2 be the number of Sylow 2-subgroups. Then by Sylow's Second Theorem (Thm 1.12, HB p39) $m_2 \equiv 1 \pmod 2$ and by Sylow's Third Theorem(b) (Thm 1.12, HB p39) $m_2|169$.

Therefore $m_2 = 1$ or 13 or 169.

Let m_{13} be the number of Sylow 13-subgroups. Then by Sylow's Second Theorem (Thm 1.12, HB p39) $m_{13} \equiv 1 \pmod{169}$ and by Sylow's Third Theorem(b) (Thm 1.12, HB p39) $m_{13}|4$.

Therefore $m_{13} = 1$. 5

(b) (i) $\mathbb{Z}_4$.

(ii) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{13} \times \mathbb{Z}_{13}$ (or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{169}$). 2

(c) $\mathbb{Z}_2 \times \mathbb{Z}_2$ 1

(d) Let N be a Sylow 13-subgroup of K. Since K is abelian, we have $kNk^{-1} = N$ for any $k \in K$. But by Sylow's Third Theorem(a) (Theorem 1.12 HB p39), all Sylow 13-subgroups are conjugate and so by the preceding observation, must be equal to N. That is, K has a unique Sylow 13-subgroup. 2

(e) (i) $ba = a^3b^{168}$ (HB p34) so $bab^{-1} = a^3b^{167}$.

(ii) $\{e, a, a^2, a^3\}$ together with a group of the form $\{e, a^3b^k, a^2, ab^k\}$ for some $k = 0, 1, 2, \ldots, 168$. 4

(f) We need to show that $ba = ab$ for every $b \in G$.

Now $ba = ab$ if and only if $bab^{-1} = a$. By Proposition 2.5(d) of Chapter 7 (HB p35) bab^{-1} also has order 2 so must be equal to a. Therefore a commutes with all elements of G and is in the centre of G. 3

17 Total

Question 8

(a) Let $r = a + b\sqrt{-10}$. From $N(a + b\sqrt{-10}) = a^2 + 10b^2 = 1$, if $b \geq 1$ then $N(r) \geq 10$. Therefore we must have $b = 0$, and hence $a = \pm 1$.

For the final conclusion, we note that if u is a unit then, by multiplicativity of N, $1 = N(1) = N(uu^{-1}) = N(u)N(u^{-1})$ shows that we must have $N(u) = 1$. So, the only possible units are ± 1, and clearly both are units. 2

(b) If $N(a + b\sqrt{-10}) = 2$ then $a^2 + 10b^2 = 2$ which implies $b = 0$ and then $a^2 = 2$ has no solution in the integers. In an entirely similar way, we find that there are no solutions to $a^2 + 10b^2 = 7$ with $a, b \in \mathbb{Z}$.

The element 2 has norm $2^2 = 4$ (so cannot be a unit by (a)). Since N is multiplicative, if $2 = rs$ for some $r, s \in R$, then either $N(r) = N(s) = 2$, or, wlog, $N(r) = 1$ and $N(s) = 4$. The first case is impossible as we just showed that there is no element with norm 2. In the second case, that $N(r) = 1$ implies that $r = \pm 1$, and we know that both of these are units. We conclude that 2 is irreducible. 3

The element 7 has norm $7^2 = 49$ (so cannot be a unit by (a)). Since N is multiplicative, if $7 = rs$ for some $r, s \in R$, then either $N(r) = N(s) = 7$, or, wlog, $N(r) = 1$ and $N(s) = 49$. The first case is impossible as we just showed that there is no element with norm 7. In the second case, that $N(r) = 1$ implies that $r = \pm 1$, and we know that both of these are units. We conclude that 7 is irreducible.

(c) If $a^2 + 10b^2 = 14$ then we have $|b| \leq 1$.

If $b = 0$ then there is no integer solution to $a^2 = 14$, so there are no elements of this form.

If $b = 1$, then $a^2 = 4$ and hence $a = \pm 2$. Similarly if $b = -1$. Thus, there are four elements with norm 14,

$$2 \pm \sqrt{-10}, -2 \pm \sqrt{-10}.$$

These elements are all irreducible, because for any proper factorisation rs the value of the norms of r and s would have to multiply to give 14. We can discount $N(r) = 1$ or $N(s) = 1$ as these are units, and so the only other possibility (up to reordering) is $N(r) = 2$, $N(s) = 7$, but we saw in part (b) that there are no elements of norm 2 or 7.

Hence all elements of norm 14 are irreducible. 3

(d) We have the factorisation $14 = 2 \cdot 7$, and we know that 2 and 7 are irreducible by part (b) so this is a product of two irreducibles.

We also have

$$14 = (2 + \sqrt{-10})(2 - \sqrt{-10})$$

and we note that the two elements on the right are irreducible by part (c).

Thus 14 can be expressed as the product of two irreducibles in two different ways. 1

(e) Since 2 divides 14, but cannot divide either of $2 \pm \sqrt{-10}$ (since $4 = N(2)$ does not divide $14 = N(2 \pm \sqrt{-10})$ it is not prime. 1

(f) (i) If $J = \langle x \rangle = \{xr : r \in R\}$, then $7 \in J$ implies $7 = xr$ for some r. But 7 is irreducible, so one of x or r must be a unit, as required.

(ii) Since $\sqrt{-10} \in J$, we know that $\sqrt{-10} = xr'$ for some r'. If x and 7 are associates, then $x = 7u$ for some unit u, and therefore $\sqrt{-10} = xr' = (7u)r' = 7(ur')$ which shows that $7 \mid \sqrt{-10}$.

(iii) If x is a unit, then $x = \pm 1$ and thus $1 = x \cdot x \in J = \langle x \rangle$. Since $1 \in J = \langle 7, \sqrt{-10} \rangle$, there must exist $a, b \in R$ such that

$$1 = 7a + \sqrt{-10}b.$$

Multiply this by $\sqrt{-10}$, to give

$$\begin{aligned} \sqrt{-10} &= 7a\sqrt{-10} + (\sqrt{-10})^2 b \\ &= 7(a\sqrt{-10} - 2b) \end{aligned}$$

which shows that $\sqrt{-10}$ is a multiple of 7.

(iv) We have $N(7) = 49$, while $N(\sqrt{-10}) = 10$, and since N is multiplicative, 7 cannot divide $\sqrt{-10}$ else 49 must divide 10. 7

17 Total

Question 9

(a) (i) For each $n \in \mathbb{N}$, the set A contains the constant function $f_n \colon [0,1] \to \mathbb{R}$ given by $f_n(x) = n$, and so for each $x \in [0,1]$, there is no $M \geq 0$ so that for every $f \in A$, we have $|f(x)| \leq M$. (Consider f_n for $n > M$.) Hence the definition of pointwise bounded does not hold. (HB p130 Definition 3.7.) 1

(ii) Fix $\varepsilon > 0$ and let $\delta = \varepsilon^2/4$, then for each $f \in A$ and each $x, y \in [0,1]$, if $|x - y| < \delta$, then

$$\begin{aligned} |f(x) - f(y)| &\leq (\sin|x| + \sin|y|)\sqrt{|x-y|}, \text{ since } f \in A \\ &\leq 2\sqrt{|x-y|}, \text{ since } x, y \in [0,1] \\ &< 2\sqrt{\delta}, \text{ since } |x-y| < \delta \\ &= \varepsilon. \end{aligned}$$ 3

Hence A is an equicontinuous set of functions. HB p130 Definition 3.8.

(Note that A is not a Lipschitz family of functions.)

(iii) B is a $d_{\max}$-closed ball and so $A \cap B$, being the intersection of two closed sets, is $d_{\max}$-closed.

$A \cap B$ is equicontinuous by using part 2 of Examples of Equicontinuous functions on p131 of HB together with the result of part (a)(ii).

$A \cap B$ is pointwise bounded because B is an example of a set of type 1 in the list of examples of pointwise bounded sets on HB p130. 3

Hence Theorem 3.12 (Arzelà-Ascoli, HB p131) implies $A \cap B$ is a $d_{\max}$-sequentially compact set.

(b) (i) We must show that d satisfies (M1)–(M3) (see HB p72, Definition 2.1).

M1 It follows immediately from the definition that for any $a, b \in \mathbb{R}$, $d(a, b) \geq 0$.

If $a = b$, then $d(a, b) = |0 - 0| + 0 = 0$.

Conversely, if $d(a, b) = 0$, then both $|a^7 - b^7| = 0$ and $d_0(a, b) = 0$, implying, since d_0 is a metric, that $a = b$. Hence (M1) holds.

M2 If $a, b \in \mathbb{R}$. then using that d_0 is a metric and properties of modulus

$$\begin{aligned} d(a, b) &= |a^7 - b^7| + d_0(a, b) \\ &= |b^7 - a^7| + d_0(b, a) \\ &= d(b, a). \end{aligned}$$

Hence (M2) holds.

M3 Let $a, b, c \in \mathbb{R}$. Then

$$\begin{aligned} d(a, c) &= |a^7 - c^7| + d_0(a, c) \\ &= |a^7 - b^7 + b^7 - c^7| + d_0(a, c) \\ &\leq |a^7 - b^7| + |b^7 - c^7| + d_0(a, c) \text{ using properties of modulus} \\ &\leq |a^7 - b^7| + |b^7 - c^7| + d_0(a, b) + d_0(b, c), \text{ since } d_0 \text{ is a metric} \\ &= |a^7 - b^7| + d_0(a, b) + |b^7 - c^7| + d_0(b, c) \\ &= d(a, b) + d(b, c). \end{aligned}$$ 7

Hence (M3) holds and we conclude that d is a metric.

(ii) Suppose that (a_n) is a d-convergent sequence with limit a, say.

Then, by definition of d-convergence HB p73 Definition 3.2, $(d(a_n, a))_{n=1}^{\infty}$ is a null sequence.

But for $n \in \mathbb{N}$, $d(a_n, a) = |a_n^7 - a^7| + d_0(a_n, a) \geq 1$ unless $a_n = a$. 3

Hence if $(d(a_n, a))_{n=1}^{\infty}$ is null, then there must be $N \in \mathbb{N}$ so that for $n > N$, $a_n = a$. In other words, (a_n) is eventually constant, as required.

17 Total

[END OF QUESTION PAPER]